

# Weight Distribution and Bounds of Turbo-Code with 3 Dimensions

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**Abstract**—To evaluate the performance of the code, it is indispensable to know how to calculate the probability of error in decoding. The weight distribution and the enumerator polynomial input-output of the code are very useful in studying the probability of error that occurs when using a maximum likelihood decoding. In this paper, we are interested the performances of turbo-code 3 dimensional (TC-3D) when it is decoded on optimizing criterion the maximum likelihood. First, we determine the mean weight distribution and the enumerator polynomial input-output of this code, which provides an upper bound on its minimum distance. Then we deduce from this distribution the bound probability of error by bit and by word on the Gaussian channel.

**Keywords**—Coding, Decoding, Enumerator Polynomial Input-Output, Minimum Distance, Probability of Error, TC- 3D, Turbo Code, Weight Distribution

**Abbreviations**—Parallel Turbo code with 2 Dimensions (TC-2D), Recursive Systematic Code (RSC), Serial Turbo Code (TC-S), Turbo-Code with 3 Dimensions (TC-3D)

## I. INTRODUCTION

MODERN wireless communication standards have already adopted various types of codes for channel coding applications. For example, the Turbo codes are used in the 3GPP Universal Mobile Telecommunications System (UMTS) [3GPP UMTS, 2007], Long Term Evolution (LTE) [3GPP LTE, 2008], Digital Video Broadcasting standard [ESTI, 2000] and the 802.16a WiMAX standard for local and metropolitan area networks [IEEE, 2003]. Furthermore, these codes are currently being proposed for next generation cellular and mobile broadband systems.

In 1993, the invention of turbo codes by Berrou et al., (1993) provided a new looks at on the existence of realistic codes approaching the limits given by information theory. The turbo-codes are based on the parallel concatenation of elementary codes linked by a random interleaver. Despite a relatively small minimum distance, their performances are separated by only 0.5 dB performance limits proposed by Shannon. In other words, although these codes are bad codes point to the usual criteria of coding theory i.e. the ratio of their minimum distance to their length tends to zero as it tends to infinity. They achieve performance very similar optimal. Distribution of this minimum distance, present the similarities with the distance distribution of a random code, which appears as an exploitation of their excellent performance [Battail, 1997].

In the future, most of digital transmission systems require low error rates which can go up to  $10^{-8}$  dB [Berrou et al., 2007]. Improving performance at very low error rates by raising the minimum hamming distance may involve at least one of these items:

- using component encoders with a larger number of states,
- devising more appropriate internal permutations,
- or increasing the dimension of the turbo code, i.e. the number of component encoders [Eirik Rosnes & Alexandre Graell i Amat, 2011].

In this work, we are interested in the third case, i.e., increasing the dimension of the turbo code.

In this paper we propose a new tree dimensional Turbo code which aims to improve probability of errors. The TC-3D we describe here is inspired from classical turbo code (TC-2D) by adding a third dimension. To evaluate the performance of this code, it is necessary to determine the average weight distribution and the enumerator polynomial input-output. So we may compute the upper bound of TC-3D minimum distance and deduce from this distribution the bound probability of error by bit and by word in the Gaussian channel. The provided results are compared to TC-2D.

The paper is structured as follows. Section II gives a short introduction to turbo codes with 2 dimensions and presents the structure of the encoding and the decoding of TC-3D. In section III presents the average weight distribution of TC-3D. In section IV, we present the polynomial input-output of this code. Section V analyzes the upper bound of

the minimum distance of the code TC-3D. The probability of errors by word and bit on a Gaussian channel are described in Section VI. The paper is summarized in the section VII.

## II. CODING STRUCTURE OF TURBO CODE WITH 3 DIMENSIONS

Generally, a Turbo code with 2 dimensions is composed of a concatenation a series [Eirik Rosnes & Alexandre Graell i Amat, 2011] or in parallel of two codes ( $C_1, C_2$ ), that is called code components, and an interleaver  $\pi$  (see figure 1). While the first code component encodes the information in the original order, the second receives the information in a permuted order. In all norms, the convolutional codes are used as component codes. For more information about TC-2D and TC-S, refer to [Benedetto & Montorsi, 1996].

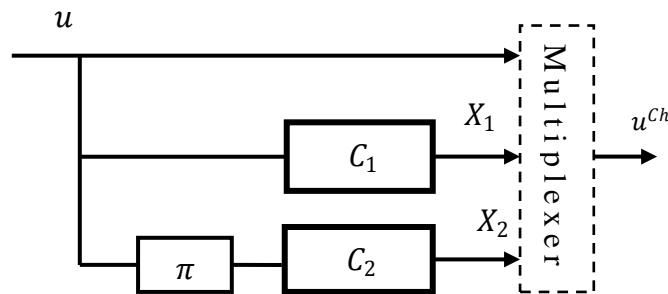


Figure 1 – Structure of Parallel Turbo Codes with 2 Dimensions

### 2.1. The Coding Structure of TC-3D

Figure 2 shows TC-3D using the three components codes  $C_1, C_2$  and  $C_3$  in parallel that will be used throughout this article to illustrate some fundamental concepts. The three codes are recursive convolutional in nature [Hagenauer & Hoeher, 1994], with a constraint length  $L = 3$  (i.e. memory=2). The overall code is a rate to equal 1/4 code with four output streams. One of the output streams is the information sequence  $u$  uncoded. The other three output streams  $X_1, X_2$  and  $X_3$  in this example are parity sequences corresponding to three codes  $C_1, C_2$  and  $C_3$ . These three parity streams would be identical if permutations  $\pi_1, \pi_2$  were not used.

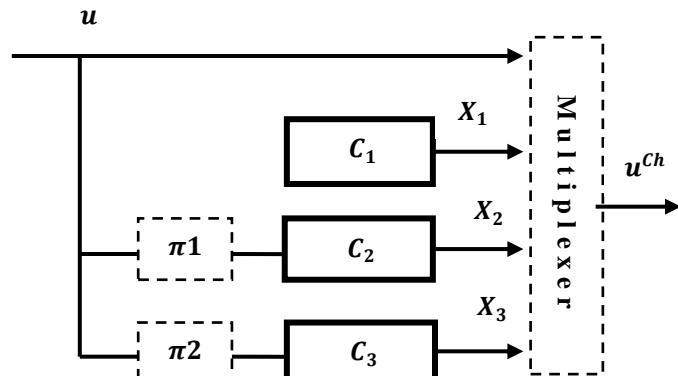


Figure 2 – TC-3D constitute from Three Codes  $C_1, C_2$  and  $C_3$  which are grouped in Parallels

The sequence  $u$  information of length  $K$  bits is encoded by TC-3D. This code is realized by the parallel concatenation of three identical coders:  $C_1 = C_2 = C_3$ . They are recursive convolutional in nature, with 8 states and the generator polynomial is:

$$G1 = [1 \ 1 \ 1]$$

$$G2 = [1 \ 0 \ 1]$$

The information bits  $u$  are encoded initially by the encoder  $C_1$  to provide the first redundancy bits  $X_1$ , then they are interleaved by  $\pi_1$  before being encoded by the encoder  $C_2$  and delivering the second redundancy  $X_2$ . The message  $u$  is then interleaved by  $\pi_2$  before being encoded by the encoder  $C_3$  and delivering the third redundancy  $X_3$ .

For the canal entrance, we send a message  $u$  and data redundancy  $X_1, X_2$  and  $X_3$  generate with three elementary codes  $C_1, C_2$  and  $C_3$ . The originality of the encoder is performing an interleaving ( $\pi_1; \pi_2$ ) on the data  $u$  before treatment  $C_2$  and  $C_3$  encoders so that errors are not corrected by the first encoder will generally be different the errors not corrected by the second and third codes.

Finally, the information sequence  $u$  and the code sequences  $X_1, X_2$  and  $X_3$  are multiplexed to form the codeword  $u^{ch}$  of length  $N$  bits, they are transmitted to the channel. Note that the total code rate of the TC-3D is  $R = K/N$ , then we can write :

$$u^{ch} = (u \ X_1 \ X_2 \ X_3) \quad (1)$$

### 2.2. Decoding of TC-3D Code

In the AWGN channel, the binary information sequence  $u^{ch} = (u \ X_1 \ X_2 \ X_3)$   $N$ -dimensional bits as input to the channel, produces a sequence  $u^{ch'} = (u' \ X'_1 \ X'_2 \ X'_3)$  the  $N$  dimensional bits, the relationship between  $u^{ch}$  and  $u^{ch'}$  for the AWGN channel is:

$$u^{ch'} = u^{ch} + B \quad (2)$$

Where  $B$  is a random sequence representing the “noise” or “error” additive

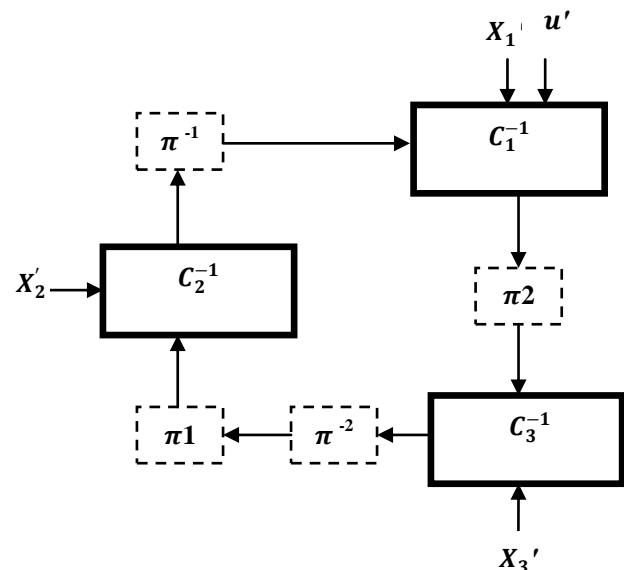


Figure 3 – Schema of TC-3D Decoder

The TC-3D can be decoded using the turbo principle. The decoder of TC-3D consists of three soft-input soft-output [Yannick Saouter, 2010] decoders  $C_1^{-1}$ ,  $C_2^{-1}$ , and  $C_3^{-1}$  corresponding to the three constituent encoders  $C_1$ ,  $C_2$ , and  $C_3$ , respectively. A decoding iteration consists of a single activation of  $C_1^{-1}$ ,  $C_3^{-1}$ , and  $C_2^{-1}$ , in this order. This process continues iteratively until the maximum number of iterations is reached or an early stopping rule criterion is fulfilled.

These three decoders exchange extrinsic information [Battail, 1997] on the systematic bits  $u'$  and the parity bits  $X_a', X_b'$  and  $X_c'$ . This extrinsic information exchange is referred as turbo principle. Figure 3 shows a possible decoder realization.

The three component decoders perform a maximum a posteriori probability (MAP) decoding on bit. They use the BCJR algorithm transformed in the logarithmic domain, the so called Log MAP algorithm [Robertson et al., 1995], to decrease the implementation complexity.

### III. AVERAGE WEIGHT DISTRIBUTION

Let  $A_1(\omega)$  be the number of code words average of  $C_1$  and the weight  $\omega$ . The average weight distribution of TC-3D is denoted  $C_1$ , is obtained by averaging over all random interleavers  $\pi_i$ ,  $i = 1, 2$  of length  $N$ .

Let  $P(\omega)$  be the probability that a sequence of weight  $\omega$  is a code word  $C$  the weight  $\omega$ , and  $A(\omega)$  the total number of code words the code  $C$  of weight  $\omega$ . There exists  $C_N^\omega$  words the weight  $\omega$  and the length  $N$ , so we have the following relationship:

$$A(\omega) = C_N^\omega P(\omega) \text{ with } C_N^\omega = \frac{N!}{(N-\omega)\omega!} \quad (3)$$

Let  $P_1(\omega)$ ,  $P_2(\omega)$  and  $P_3(\omega)$  be the probability that a sequence of weights  $\omega$  is a codeword  $C_1, C_2, C_3$  respectively. The codes  $C_2$  and  $C_3$  being permuted versions of the code  $C_1$ , probabilities  $P_1(\omega)$ ,  $P_2(\omega)$  and  $P_3(\omega)$  are identical and equal to  $P_1(\omega)$ , so the probability  $P(\omega)$  can be written:

$$P(\omega) = P_1(\omega) \times P_2(\omega) \times P_3(\omega) \quad (4)$$

Of over:

$$A_1(\omega) = C_N^\omega P_1(\omega) \quad (5)$$

The mean distribution of the weight of TC-3D code is given by the following relationship:

$$A(\omega) = \frac{(A_1(\omega))^3}{(C_N^\omega)^2} \quad (6)$$

Weight distribution  $A_1(\omega)$  of  $C_1$  convolutional code can be calculated using the transition matrix  $T$  raised to the power  $W$ , where  $W$  is the number of branches in the trellis code [Divsalar et al., 1995].

The previous formula was applied to a code of length  $N = 200$  constructed from two recursive systematic codes (RSC) of rate  $1/2$  and generator polynomial  $G1$  and  $G2$ , for calculate the weight distribution  $A(\omega)$ :

$$G1 = [1 \ 1 \ 1]$$

$$G2 = [1 \ 0 \ 1]$$

Table 1 – Average Weight Distribution of TC-3D

w	A(w)	w	A(w)	w	A(w)	w	A(w)
5	1,24E-81	12	3,00E-72	19	8,75E-65	26	1,49E-58
6	4,05E-80	13	4,34E-71	20	7,92E-64	27	3,06E+31
7	1,12E-78	14	5,79E-70	21	6,79E-63	28	1,89E+32
8	2,70E-77	15	7,18E-69	22	5,52E-62	29	1,12E+33
9	5,77E-76	16	8,30E-68	23	4,28E-61	30	6,40E+33
10	1,10E-74	17	8,99E-67	24	3,15E-60		
11	1,90E-73	18	9,14E-66	25	2,22E-59		

This weight distribution is compared to turbo code with two dimensions for  $N=200$  in Figure 4. The two distributions are shown in Figure 4.

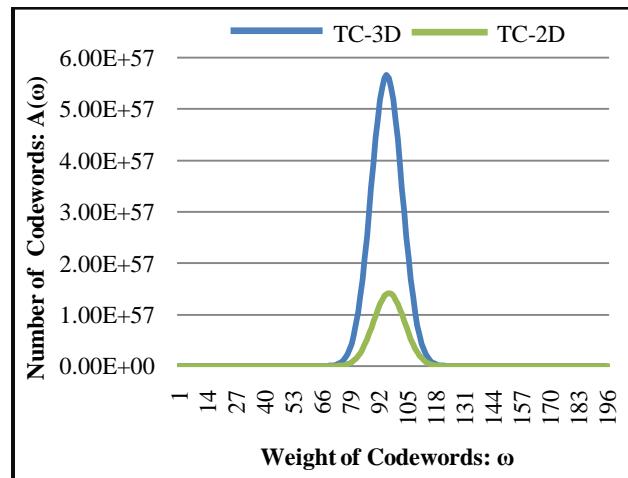


Figure 4 – Comparison of Weight Distribution TC-3D Code and the Weight Distribution of a Turbo Code in Two Dimensions for  $N=200$

### IV. POLYNOMIAL ENUMERATOR INPUT-OUTPUT OF TC-3D

To simplify the study, we assume throughout this subparagraph the three coders  $C_1, C_2$  and  $C_3$  are RSC and identical i.e.  $C_1 = C_2 = C_3$ . Their rate is  $R_1 = R_2 = R_3 = 1/2$ . The rate of the resulting TC-3D is equal to  $R=1/4$ . Trellis termination is neglected, and we denote by  $N_\pi$  the size of the interleaver. The enumerator polynomial input-output code  $C_1$  wrote:

$$A^{C_1}(I, Z) = \sum_{i=i_{\min}}^{N_\pi} \sum_{z=z_{\min}}^{N_\pi} A_{i,z}^{C_1} I^i Z^z \quad (7)$$

Where  $A_{i,z}^{C_1}$  is the number of codewords  $C_1$  corresponding to a weight of information  $i$  and the weight parity  $z$ . The polynomial  $A^{C_1}(I, Z)$  does not include the word 0, ..., 0. For a RSC code which trellis is closed, the minimum weight input is  $i_{\min}$ . The minimum weight output  $z_{\min}$  does not necessarily correspond to minimum weight input  $i_{\min}$ ; if we note  $z'_{\min}$  the weight of the parity corresponding to the weight input  $i_{\min}$ , then in general:

$$z_{\min} \leq z'_{\min}$$

The polynomial  $A^{C_1}(I, Z)$  can also be written as follows:

$$A^{C_1}(I, Z) = \sum_{i=i_{\min}}^{N_\pi} I^i A^{C_1}(i, Z) \quad (8)$$

With

$$A^{C_1}(i, Z) = \sum_{z=z_{\min}}^{N_\pi} A_{i,z}^{C_1} Z^z \quad (9)$$

Where  $A^{C_1}(i, Z)$  is called conditional weight enumerator polynomial because the weight of the input  $i$  is fixed. A method for calculating this polynomial is presented in Benedetto & G. Montorsi (1996) and Battail (1997).

The conditional weight enumerator polynomial conditional TC-3D considered is calculated by averaging over all possible interleavers. The interleaver TC-3D is random; three encoders' inputs are independent. Therefore it is easily verified:

$$A^C(i, Z) = \frac{(A^{C_1}(i, Z))^3}{(C_N^i)^2} \quad (10)$$

## V. SUPERIOR BOUND OF THE MINIMUM DISTANCE OF TC-3D CODE

The average weight distribution of TC-3D code provides an upper bound  $\rho$  of its minimum distance  $d_{H \min}$  for a length code  $N$  fixed. In fact,

$$\Pr(d_{H \min} \leq \rho) = \Pr(\exists v \in (GF(2))^N, \omega_v \leq \rho, v \in C) \quad (11)$$

Where  $\omega_v$  denotes the weight of the sequence  $v$ . The terminal of the union applied to this equality gives:

$$\Pr(d_{H \min} \leq \rho) \leq \sum_{i=1}^{\rho} \Pr(\exists v \in (GF(2))^N, v \in C, \omega_v = i) \quad (12)$$

But there are  $M = C_N^i$  the sequences of weight  $i$  in  $(GF(2))^N$  denoted  $s_1, \dots, s_M$  thus:

$$\Pr(d_{H \min} \leq \rho) \leq \sum_{i=1}^{\rho} \Pr((s_1 \in C, \omega_{s_1} = i) \cup (s_2 \in C, \omega_{s_2} = i) \cup \dots \cup (s_M \in C, \omega_{s_M} = i)) \quad (13)$$

Let,

$$\Pr(d_{H \min} \leq \rho) \leq \sum_{i=1}^{\rho} \sum_{j=1}^M \Pr(s_j \in C, \omega_{s_j} = i) \quad (14)$$

The probability  $\Pr(s_j \in C, \omega_{s_j} = i)$  is by definition to equal to the probability  $P(i), \forall j$ , hence:

$$\Pr(d_{H \min} \leq \rho) \leq \sum_{i=1}^{\rho} M P(i) \quad (15)$$

$$\leq \sum_{i=1}^{\rho} C_N^i P(i) \quad (16)$$

$$\leq \sum_{i=1}^{\rho} A(i) \quad (17)$$

Consequently, an upper bound is given by  $\rho$ , the smallest integer satisfying:

$$\sum_{w=1}^{\rho} A(w) = 1 \quad (18)$$

Table 2 – Presents some Values of  $\rho$  for the TC-3D Code for Different Length  $N$

Length N	Superior Bound $\rho$
100	36,840
200	46,416
300	53,133
400	58,480
500	62,996
600	66,943
700	70,473
800	73,681

Figure 5 represents evolution the superior bound  $\rho$  of the minimum distance for lengths codes from  $N=100$  to  $N=800$ , total rate  $R = 1/4$  and pseudo-random interleaving, in the three following cases:

- TC-3D
- Parallel Turbo code with 2 dimension (TC-2D)
- Serial Turbo Code ( TC-S)

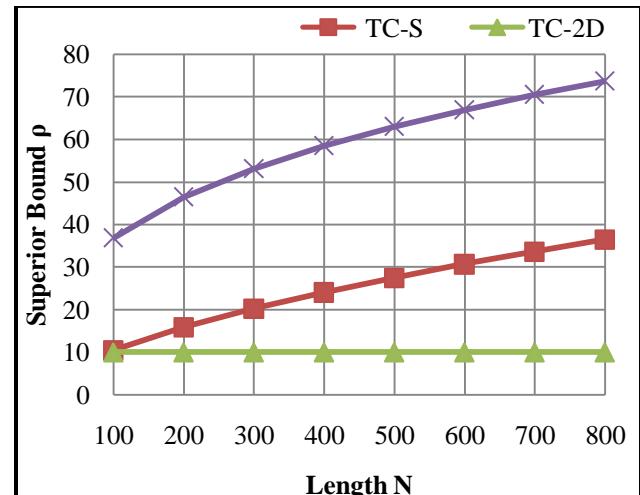


Figure 5 – Evolution the Superior Bound  $\rho$  of Minimum Distance

More than the same values of the superior bound  $\rho$ , it is interesting to note on this figure, the nature of the evolution of  $\rho$  as a function of  $N$ . Indeed, in the case of TC-3D this evolution is higher than the turbo-code parallel and serial with 2 dimensions. For the turbo code with two dimensions, distance varies very little as a function the length  $N$  of the code. This behavior is similar to the results of the minimum distance presented by Kahale & Urbanke (1998).

## VI. PROBABILITY OF ERROR ON THE GAUSSIAN CHANNEL

The average weight distribution of the TC-3D code can also enable us to deduce bounds for the error probability of this code on the AWGN channel. Indeed, the interleavers in TC-3D code act on all the bits coded such that all bits coded are equally protected. The probability of error by word  $P_{e_W}$  and

bit  $P_{e_b}$  are then obtained by applying the bound of the union and increased by:

$$P_{e_W} \leq \sum_{\omega=d_1}^N A(\omega) Q\left(\sqrt{R\omega \frac{2E_b}{N_0}}\right) \quad (19)$$

$$P_{e_b} \leq \sum_{\omega=d_1}^N \frac{\omega}{N} A(\omega) Q\left(\sqrt{R\omega \frac{2E_b}{N_0}}\right) \quad (20)$$

Where  $E_b/N_0$  is signal-to-noise ratio average by bit and  $Q(x)$  denotes the error function. Figure 6 shows the probability of error by bit obtained after decoding the code TC-3D, the rate  $R = 1/4$  constructed from convolutional code for  $K=1024$  bits,  $K=3072$  bits and  $K=6148$  bits.

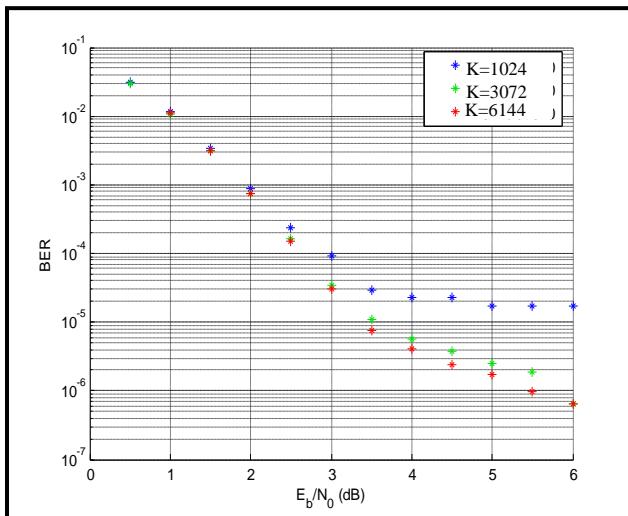


Figure 6 – Probability of Error by Bit for  $K=1024$  bits,  $K=3072$  bits and  $K=6148$  bits

Let us note that bounds more precise at low signal-to-noise ratios can be considered to estimate the probability of error by word and bit.

## VII. CONCLUSION

To estimate the performance of a code, it is necessary to have information on its minimum distance and the probability of error. The weight distribution and enumerator polynomial of the code are very useful for studying these performances. In this paper we have calculated the weight distribution and enumerator polynomial input-output of TC-3D to evaluate their minimum distance and their error probability. The results found are compared to the turbo-code with 2 dimensions. We deduce from these results (Figure 4 and Figure 5) that the performances of TC-3D are better than those of the turbo-code with 2 dimensions. In the future work, we will study the relationship between the inputs and the outputs of our code, i.e. the transfer function.

## REFERENCES

- [1] C. Berrou, A. Glavieux & P. Thitimajshima (1993) “Near-Shannon Limit Error-Correcting Coding and Decoding: Turbo-Codes”, *IEEE International Conference on Communications* (ICC '93), Geneva. Technical Program, Conference Record, Pp.1064-1070.
- [2] J. Hagenauer & P. Hoeher (1994), “A Viterbi Algorithm with Soft-Decision Outputs and its Applications”, *Proceedings /IEEE Globecom Conference*, Vol.3, Pp. 1680-1686.
- [3] P. Robertson, E. Villebrun & P. Hoeher (1995), “A Comparison of Optimal and Sub-Optimal MAP Decoding Algorithms Operating in the Log-Domain”, *Proceeding/ International Conference on Communications (ICC '95)*, Vol.2, Pp. 1009-1013.
- [4] D. Divsalar, S. Dolinar & F. Pollara (1995), “Transfer Function Bounds on the Performance of Turbo Codes”, *TDA Progress Report 42-122*, Pp. 44-55.
- [5] S. Benedetto & G. Montorsi (1996), “Unveiling Turbo Codes: Some Results on Parallel Concatenated Coding Schemes”, *IEEE Transaction on Information Theory*, Vol. 42, No. 2, Pp. 409-429.
- [6] S. Benedetto & G. Montorsi (1996), “Serial Concatenation of Block and Convolutional Code”, *Electronic Letters*, Vol.32, No.10, Pp.887-888.
- [7] G. Battail (1997), “Théorie d'Information”, Masson, Paris.
- [8] N. Kahale & R. Urbanke (1998), “On the Minimum Distance of Parallel and Serially Concatenated Code”, *Proceeding/IEEE International Symposium on Information Theory*, Pp.31.
- [9] ESTI (2000), “Digital Video Broadcasting (DVB); Interaction Channel for Satellite Distribution Systems”, *ETSI EN 301 790*, V1.2.2, Pp.20-24
- [10] IEEE (2003), “IEEE Standard for Local and Metropolitan Area Networks”, *IEEE 802.16a*.
- [11] 3GPP UMTS (2007), “General UMTS Architecture”, *3GPP TS 23.101 version 7.0.0*.
- [12] C. Berrou, A. Graell i Amat, Y. Ould Cheikh Mouhamedou, C. Douillard & YSaouter (2007), “Adding a Rate-1 Third Dimension to Turbo Codes” *Proceedings/ IEEE Information Theory Workshop (ITW '07)*, Pp.156–161.
- [13] 3GPP LTE (2008), “Evolved Universal Terrestrial Radio Access (EUTRA) and Evolved Universal Terrestrial Radio Access Network (EUTRAN)”, *3GPP TS 36.300*.
- [14] Yannick Saouter (2010), “Decoding”, URL: <http://www.techniques-ingeneur.fr/base-documentaire/technologies-de-l-information-th9/traitement-du-signal-bases-theoriques-42295210/turbocodes-realisations-et-perspectives-te5260/decodage-te5260niv10003.html>.
- [15] Eirik Rosnes & Alexandre Graell i Amat (2011), “Performance Analysis of 3-Dimensional Turbo Codes”, *IEEE Transaction on Information Theory*, Vol. 57, No. 6, Pp. 3707-3720.



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